Lecture 7 (10/8/21)

- Finish discussion of uniform convergence from Lecture 6 notes.

Power Series.

Def. (1) A series of complex numbers $a_n \in \mathbb{C}$, $\sum_{n=0}^{\infty} a_n$, converges to $s \in \mathbb{C}$ if the seq. of partial sums $s_n = \sum_{k=0}^{n} a_k$ converges to s.

Rem. Since $\mathbb{C}$ is complete, suffices that $\{s_n\}_{}^{\infty}$ is a Cauchy seq.

(2) $\sum_{n=0}^{\infty} a_n$ conv. absolutely if $\sum_{n=0}^{\infty} |a_n|$ converges to $\sigma \in [0,\infty)$. Rem. abs. conv. $\Rightarrow$ conv. by completeness and conv. $\Rightarrow |a_n| \to 0$.

(3) A power series centered at $a \in \mathbb{C}$ is a series $\sum_{n=0}^{\infty} a_n (z-a)^n$, $a_n \in \mathbb{C}$, where $z = x+iy \in \mathbb{C}$ is a variable.

Ex. (1) The geometric series $\sum_{n=0}^{\infty} z^n$ converges to $\frac{1}{1-z}$ for $|z|<1$.

Thm Let $\sum_{n=0}^{\infty} a_n (z-a)^n$ be a power series and $R^{-1} = \limsup_{n\to\infty} |a_n|^{1/n}$, where $R=0$ and $R=\infty$ are allowed.

(a) The series conv. absolutely for $|z-a|<R$.

(b) —— " — diverges if $|z-a|>R$.

(c) For $0<r<R$, the series converges abs. and unif. in $|z-a|\le r$.

Pf. Let us assume $0<R<\infty$, leaving $R=0,\infty$ to DIY. First, (c): Fix $r<R$ and pick $r<\rho<R$. Then $\frac{1}{R}<\frac{1}{\rho}$, so $\exists\, N$ s.t. $|a_n|^{1/n}\le\frac{1}{\rho}$ for $n\ge N$. But then if $N\le n\le m$, $|z-a|\le r$,

$$\left|\sum_{k=n}^{m} a_k (z-a)^k\right| \le \sum_{k=n}^{m} |a_k|\cdot r^k \le \sum_{k=n}^{m}\left(\frac{r}{\rho}\right)^k.$$

Since $\frac{r}{\rho}<1$, $\sum_{k=n}^{m}\left(\frac{r}{\rho}\right)^k \le \left(\frac{r}{\rho}\right)^n \frac{1}{1-r/\rho} \to 0$

as $n\to\infty$. By completeness, we conclude there is function $f(z)$ in $|z-a|\le r$ s.t. $\sum_{n=0}^{\infty} a_n(z-a)^n \to f(z)$, the convergence is absolute and, by Weierstrass M Thm, uniform.

Clearly, (c) $\Rightarrow$ (a).

For (b), note that if $|z_0 - a| = r_0 > R$, then we can pick $R < \rho_0 < r_0$. Since $\limsup_{n\to\infty} |a_n|^{1/n} = \frac{1}{R}$

$\exists$ subseq. a_{n_k} s.t. $|a_{n_k}|^{1/n_k} \to \frac{1}{R}$. Then, for large k, $|a_{n_k}|^{1/n_k} > \frac{1}{\rho_0} \Rightarrow |a_{n_k}(z_0 - a)^{n_k}|$

$> \left(\frac{r_0}{\rho_0}\right)^{n_k} \to \infty$ since $\frac{r_0}{\rho_0} > 1$. Thus,

$\sum_{n=0}^{\infty} a_n(z-a)^n$ cannot converge as terms $\not\to 0$. ☒

Rem. In (c), we proved the stronger statement that for $|z-a| \le r < R$, the tail of series satisfies

$$\sum_{k=n}^{m} |a_k(z-a)^k| \le \sum_{k=n}^{m} M_k \to 0$$

as $n, m \to \infty$. Such convergence of series is called normal (not in Conway) and implies abs. and unif. convergence.

__Def__ (1) The number $R \in [0,\infty]$ in Thm1 is called __radius of convergence__ of $\sum_{n=0}^{\infty} a_n (z-a)^n$.